Searching for patterns in certain random lattice graphs

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1 Background

We consider the (somewhat popular) two person complete information game which is started on an empty finite square lattice (in general, say $n \times n$) of points. Players take turns to make moves,- drawing lines between two adjacent unconnected points. If a player completes a unit square by making a move, he 'owns' that square and gains an extra move (and this can go on). When the whole grid is filled, the one who owns more no. of squares wins.

The initial interest was to get some idea about the winning strategy, if any. It seems that as n grows, getting a general strategy would be difficult. Another approach is interesting. The practical initial strategy of any player for his move would be to ensure that his opponent can't gain any square in his next move. However, obviously, at some point of playing, one among them would find himself in a position that whatever move he gives, his opponent will gain at least one square in the next move. We call these type of positions in the grid **critical positions**. We can also check that if we can characterize the critical positions in some way - maybe some equivalence relations, maybe they can be changed into one another by some given set of standard 'operations', and so on.

2 General Strategies

Here are some general strategies that *may* come useful in course of work.

1. **Converting any position in the grid into an equivalent lattice animal :** We replace each square by a point, each vertical line between two squares in the grid by a horizontal line between the corresponding points, and similarly, each horizontal line by an equivalent vertical one (the external lines are assumed to connect to a common point). For example, the following critical position and critical lattice graph are equivalent :

2. **Connection matrices :** The horizontal and vertical connections form two binary matrices, say H and V, respectively. In the equivalent $n \times n$ point-grid setup, H is $n \times (n+1)$ and V is $(n+1) \times n$.For example, in the above point-grid, the matrices are -

   - Here also is an interesting proposition, made from observing randomly (by hand) generated critical positions :

     Proposition 1 *For critical positions, HV is singular.*

3 An alternative problem

3.1 Approach

As an alternative problem we first consider the 'non-practical' scenario, where players don't play to win. In this case, the first square can be completed by a minimum of 4 lines, and can be dragged till our critical position. Naturally, we can look for the underlying distribution from simulated data. We wrote a C program that, given n puts a square randomly in the $n \times n$ grid, and randomly keeps on putting lines in the grid until a plausible exit condition is satisfied. For making things scale-free, we look at the distribution of number of lines required to complete first square *divided by* n^2.

In the same way we can look for the distribution of (proportion of) lines required to get 2nd square *after getting the first square*, and in general (proportion of) lines needed to get r-th square after completing $(r-1)^{th}$ square. We modified our program to suit this need, and simulated 10000 datapoints each for $n = 10, 15, ..., 245$ and $r = 1, 2, 5, 7, 10, 12, 15, 18, 21, 25, 27, 30$.

3.2 Results

1. The data were plotted for different n and r and from the logarithm plot it seems that the distributions are exponential (β distribution with suitable parameters is also a possibility, because of the nature of the data, i.e. they can only take values in $[0, 1]$) and they approach the random variable having probability 1 at 0 as n and r grow to infinity.

2. The expected no. of lines to complete r-th square after getting $(r-1)^{th}$ square is expected to be a function of n and r, say $E(n, r)$. So, for given n, the expectation is a function of r, say $E_n(r)$. For fixed n, the data means were plotted vs. r, which naturally show a decreasing trend. Regression

of the double log data (i.e. $\log(E_n(r))$ vs. $\log r$) was found significant (all p-values are of order $\leq 10^{-8}$), indicating a power law structure, i.e.

$$E_n(r) = A_0 r^{a_1}$$

3. To get $E(n, r)$ from here, A_0 and a_1 as functions of n have to be determined. Quadratic regression of A_0 on n seems a good fit. From the plot of a_1 vs. n, it seems that the values decrease to a limit as $n \to \infty$. Proposal of a plausible model for fitting the plot is expected.

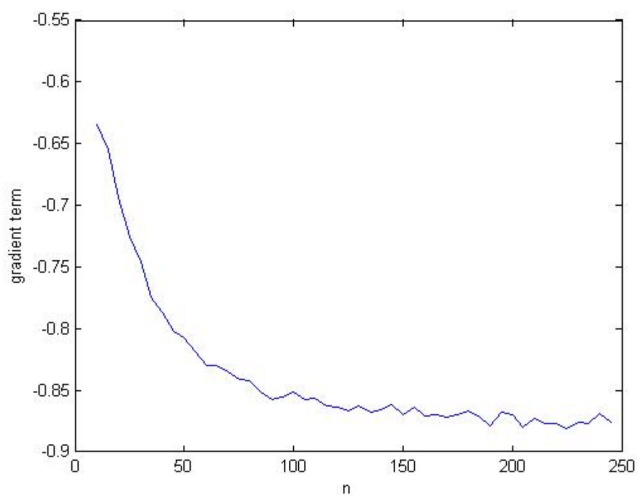

Figure 1: Plot of gradient term (a_1) vs. n